

Theoretical Computer Science Cheat Sheet

| Definitions | | Series | |
|--|---|---|---|
| $f(n) = O(g(n))$ | iff \exists positive c, n_0 such that $0 \leq f(n) \leq cg(n) \forall n \geq n_0$. | $\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$ | |
| $f(n) = \Omega(g(n))$ | iff \exists positive c, n_0 such that $f(n) \geq cg(n) \geq 0 \forall n \geq n_0$. | In general: | |
| $f(n) = \Theta(g(n))$ | iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. | $\sum_{i=1}^n i^m = \frac{1}{m+1} \left[(n+1)^{m+1} - 1 - \sum_{i=1}^n ((i+1)^{m+1} - i^{m+1} - (m+1)i^m) \right]$ | |
| $f(n) = o(g(n))$ | iff $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$. | $\sum_{i=1}^{n-1} i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}.$ | |
| $\lim_{n \rightarrow \infty} a_n = a$ | iff $\forall \epsilon > 0, \exists n_0$ such that $ a_n - a < \epsilon, \forall n \geq n_0$. | Geometric series: | |
| $\sup S$ | least $b \in \mathbb{R}$ such that $b \geq s, \forall s \in S$. | $\sum_{i=0}^n c^i = \frac{c^{n+1} - 1}{c - 1}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1 - c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1 - c}, \quad c < 1,$ | |
| $\inf S$ | greatest $b \in \mathbb{R}$ such that $b \leq s, \forall s \in S$. | $\sum_{i=0}^n ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad c < 1.$ | |
| $\liminf_{n \rightarrow \infty} a_n$ | $\lim_{n \rightarrow \infty} \inf \{a_i \mid i \geq n, i \in \mathbb{N}\}.$ | Harmonic series: | |
| $\limsup_{n \rightarrow \infty} a_n$ | $\lim_{n \rightarrow \infty} \sup \{a_i \mid i \geq n, i \in \mathbb{N}\}.$ | $H_n = \sum_{i=1}^n \frac{1}{i}, \quad \sum_{i=1}^n iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$ | |
| $\binom{n}{k}$ | Combinations: Size k subsets of a size n set. | $\sum_{i=1}^n H_i = (n+1)H_n - n, \quad \sum_{i=1}^n \binom{i}{m} H_i = \binom{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right).$ | |
| $[n_k]$ | Stirling numbers (1st kind): Arrangements of an n element set into k cycles. | 1. $\binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad 2. \sum_{k=0}^n \binom{n}{k} = 2^n, \quad 3. \binom{n}{k} = \binom{n}{n-k},$ | |
| $\{n_k\}$ | Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets. | 4. $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}, \quad 5. \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$ | |
| $\langle n_k \rangle$ | 1st order Eulerian numbers: Permutations $\pi_1 \pi_2 \dots \pi_n$ on $\{1, 2, \dots, n\}$ with k ascents. | 6. $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}, \quad 7. \sum_{k=0}^n \binom{r+k}{k} = \binom{r+n+1}{n},$ | |
| $\llbracket n_k \rrbracket$ | 2nd order Eulerian numbers. | 8. $\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}, \quad 9. \sum_{k=0}^n \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n},$ | |
| C_n | Catalan Numbers: Binary trees with $n+1$ vertices. | 10. $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}, \quad 11. \left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1,$ | |
| 14. $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!,$ | 15. $\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!H_{n-1},$ | 16. $\begin{bmatrix} n \\ n \end{bmatrix} = 1,$ | 17. $\begin{bmatrix} n \\ k \end{bmatrix} \geq \begin{Bmatrix} n \\ k \end{Bmatrix},$ |
| 18. $\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$ | 19. $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2},$ | 20. $\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} = n!,$ | 21. $C_n = \frac{1}{n+1} \binom{2n}{n},$ |
| 22. $\langle \begin{matrix} n \\ 0 \end{matrix} \rangle = \langle \begin{matrix} n \\ n-1 \end{matrix} \rangle = 1,$ | 23. $\langle \begin{matrix} n \\ k \end{matrix} \rangle = \langle \begin{matrix} n \\ n-1-k \end{matrix} \rangle,$ | 24. $\langle \begin{matrix} n \\ k \end{matrix} \rangle = (k+1) \langle \begin{matrix} n-1 \\ k \end{matrix} \rangle + (n-k) \langle \begin{matrix} n-1 \\ k-1 \end{matrix} \rangle,$ | |
| 25. $\langle \begin{matrix} 0 \\ k \end{matrix} \rangle = \begin{cases} 1 & \text{if } k=0, \\ 0 & \text{otherwise} \end{cases}$ | 26. $\langle \begin{matrix} n \\ 1 \end{matrix} \rangle = 2^n - n - 1,$ | 27. $\langle \begin{matrix} n \\ 2 \end{matrix} \rangle = 3^n - (n+1)2^n + \binom{n+1}{2},$ | |
| 28. $x^n = \sum_{k=0}^n \langle \begin{matrix} n \\ k \end{matrix} \rangle \binom{x+k}{n},$ | 29. $\langle \begin{matrix} n \\ m \end{matrix} \rangle = \sum_{k=0}^m \binom{n+1}{k} (m+1-k)^n (-1)^k,$ | 30. $m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \sum_{k=0}^n \langle \begin{matrix} n \\ k \end{matrix} \rangle \binom{k}{n-m},$ | |
| 31. $\langle \begin{matrix} n \\ m \end{matrix} \rangle = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \binom{n-k}{m} (-1)^{n-k-m} k!,$ | 32. $\llbracket \begin{matrix} n \\ 0 \end{matrix} \rrbracket = 1,$ | 33. $\llbracket \begin{matrix} n \\ n \end{matrix} \rrbracket = 0$ for $n \neq 0,$ | |
| 34. $\llbracket \begin{matrix} n \\ k \end{matrix} \rrbracket = (k+1) \llbracket \begin{matrix} n-1 \\ k \end{matrix} \rrbracket + (2n-1-k) \llbracket \begin{matrix} n-1 \\ k-1 \end{matrix} \rrbracket,$ | 35. $\sum_{k=0}^n \llbracket \begin{matrix} n \\ k \end{matrix} \rrbracket = \frac{(2n)^n}{2^n},$ | | |
| 36. $\left\{ \begin{matrix} x \\ x-n \end{matrix} \right\} = \sum_{k=0}^n \llbracket \begin{matrix} n \\ k \end{matrix} \rrbracket \binom{x+n-1-k}{2n},$ | 37. $\left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} = \sum_{k=0}^n \left\{ \begin{matrix} k \\ m \end{matrix} \right\} (m+1)^{n-k},$ | | |

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Identities Cont.

$$\begin{aligned}
 38. \quad \binom{n+1}{m+1} &= \sum_k \binom{n}{k} \binom{k}{m} = \sum_{k=0}^n \binom{k}{m} n^{n-k} = n! \sum_{k=0}^n \frac{1}{k!} \binom{k}{m}, & 39. \quad \begin{bmatrix} x \\ x-n \end{bmatrix} &= \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \binom{x+k}{2n}, \\
 40. \quad \left\{ \begin{matrix} n \\ m \end{matrix} \right\} &= \sum_k \binom{n}{k} \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\} (-1)^{n-k}, & 41. \quad \begin{bmatrix} n \\ m \end{bmatrix} &= \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \binom{k}{m} (-1)^{m-k}, \\
 42. \quad \left\{ \begin{matrix} m+n+1 \\ m \end{matrix} \right\} &= \sum_{k=0}^m k \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\}, & 43. \quad \begin{bmatrix} m+n+1 \\ m \end{bmatrix} &= \sum_{k=0}^m k(n+k) \begin{bmatrix} n+k \\ k \end{bmatrix}, \\
 44. \quad \binom{n}{m} &= \sum_k \binom{n+1}{k+1} \binom{k}{m} (-1)^{m-k}, & 45. \quad (n-m)! \binom{n}{m} &= \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} (-1)^{m-k}, \quad \text{for } n \geq m, \\
 46. \quad \left\{ \begin{matrix} n \\ n-m \end{matrix} \right\} &= \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \begin{bmatrix} m+k \\ k \end{bmatrix}, & 47. \quad \begin{bmatrix} n \\ n-m \end{bmatrix} &= \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \left\{ \begin{matrix} m+k \\ k \end{matrix} \right\}, \\
 48. \quad \left\{ \begin{matrix} n \\ \ell+m \end{matrix} \right\} \binom{\ell+m}{\ell} &= \sum_k \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} \left\{ \begin{matrix} n-k \\ m \end{matrix} \right\} \binom{n}{k}, & 49. \quad \begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell} &= \sum_k \begin{bmatrix} k \\ \ell \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} \binom{n}{k}.
 \end{aligned}$$

Trees

Every tree with n vertices has $n - 1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are d_1, \dots, d_n :

$$\sum_{i=1}^n 2^{-d_i} \leq 1,$$

and equality holds only if every internal node has 2 sons.

Recurrences

Master method:

$$T(n) = aT(n/b) + f(n), \quad a \geq 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then

$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then

$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \leq cf(n)$ for large n , then

$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence

$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two.

Let $t_i = \log_2 T_i$. Then we have

$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get

$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find

$$u_{i+1} = \frac{1}{2} + u_i, \quad u_1 = \frac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$.

Summing factors (example): Consider the following recurrence

$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side

$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"

$$\begin{aligned}
 1(T(n) - 3T(n/2)) &= n \\
 3(T(n/2) - 3T(n/4)) &= n/2 \\
 \vdots & \\
 3^{\log_2 n - 1}(T(2) - 3T(1)) &= 2
 \end{aligned}$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get

$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\frac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have

$$\begin{aligned}
 n \sum_{i=0}^{m-1} c^i &= n \left(\frac{c^m - 1}{c - 1} \right) \\
 &= 2n(c^{\log_2 n} - 1) \\
 &= 2n(c^{(k-1)\log_2 n} - 1) \\
 &= 2n^k - 2n,
 \end{aligned}$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider

$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that

$$T_{i+1} = 1 + \sum_{j=0}^i T_j.$$

Subtracting we find

$$\begin{aligned}
 T_{i+1} - T_i &= 1 + \sum_{j=0}^i T_j - 1 - \sum_{j=0}^{i-1} T_j \\
 &= T_i.
 \end{aligned}$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i .
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i .

Example:

$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:

$$\sum_{i \geq 0} g_{i+1} x^i = \sum_{i \geq 0} 2g_i x^i + \sum_{i \geq 0} x^i.$$

We choose $G(x) = \sum_{i \geq 0} x^i g_i$. Rewrite in terms of $G(x)$:

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \geq 0} x^i.$$

Simplify:

$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:

$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:

$$\begin{aligned}
 G(x) &= x \left(\frac{2}{1-2x} - \frac{1}{1-x} \right) \\
 &= x \left(2 \sum_{i \geq 0} 2^i x^i - \sum_{i \geq 0} x^i \right) \\
 &= \sum_{i \geq 0} (2^{i+1} - 1) x^{i+1}.
 \end{aligned}$$

So $g_i = 2^i - 1$.

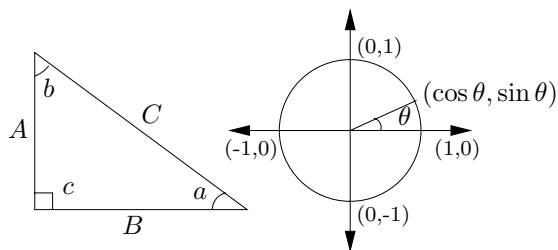
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$$\pi \approx 3.14159, \quad e \approx 2.71828, \quad \gamma \approx 0.57721, \quad \phi = \frac{1+\sqrt{5}}{2} \approx 1.61803, \quad \hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -0.61803$$

| i | 2^i | p_i | General | Probability |
|---|---------------|-------|--|---|
| 1 | 2 | 2 | Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$): | Continuous distributions: If |
| 2 | 4 | 3 | $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30},$ | $\Pr[a < X < b] = \int_a^b p(x) dx,$ |
| 3 | 8 | 5 | $B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}.$ | then p is the probability density function of X . If |
| 4 | 16 | 7 | Change of base, quadratic formula: | $\Pr[X < a] = P(a),$ |
| 5 | 32 | 11 | $\log_b x = \frac{\log_a x}{\log_a b}, \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$ | then P is the distribution function of X . If P and p both exist then |
| 6 | 64 | 13 | Euler's number e : | $P(a) = \int_{-\infty}^a p(x) dx.$ |
| 7 | 128 | 17 | $e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$ | Expectation: If X is discrete |
| 8 | 256 | 19 | $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$ | $E[g(X)] = \sum_x g(x) \Pr[X = x].$ |
| 9 | 512 | 23 | $\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}.$ | If X continuous then |
| 10 | 1,024 | 29 | $\left(1 + \frac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$ | $E[g(X)] = \int_{-\infty}^{\infty} g(x)p(x) dx = \int_{-\infty}^{\infty} g(x) dP(x).$ |
| 11 | 2,048 | 31 | Harmonic numbers: | Variance, standard deviation: |
| 12 | 4,096 | 37 | $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{7129}{2520}, \dots$ | $\text{VAR}[X] = E[X^2] - E[X]^2,$ |
| 13 | 8,192 | 41 | $\ln n < H_n < \ln n + 1,$ | $\sigma = \sqrt{\text{VAR}[X]}.$ |
| 14 | 16,384 | 43 | $H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$ | For events A and B : |
| 15 | 32,768 | 47 | Factorial, Stirling's approximation: | $\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$ |
| 16 | 65,536 | 53 | $1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \dots$ | $\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$ |
| 17 | 131,072 | 59 | $n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$ | iff A and B are independent. |
| 18 | 262,144 | 61 | Ackermann's function and inverse: | $\Pr[A B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$ |
| 19 | 524,288 | 67 | $a(i, j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$ | For random variables X and Y : |
| 20 | 1,048,576 | 71 | $\alpha(i) = \min\{j \mid a(j, j) \geq i\}.$ | $E[X \cdot Y] = E[X] \cdot E[Y],$ |
| 21 | 2,097,152 | 73 | Binomial distribution: | if X and Y are independent. |
| 22 | 4,194,304 | 79 | $\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \quad q = 1 - p,$ | $E[X + Y] = E[X] + E[Y],$ |
| 23 | 8,388,608 | 83 | $E[X] = \sum_{k=1}^n k \binom{n}{k} p^k q^{n-k} = np.$ | $E[cX] = c E[X].$ |
| 24 | 16,777,216 | 89 | Poisson distribution: | Bayes' theorem: |
| 25 | 33,554,432 | 97 | $\Pr[X = k] = \frac{e^{-\lambda} \lambda^k}{k!}, \quad E[X] = \lambda.$ | $\Pr[A_i B] = \frac{\Pr[B A_i] \Pr[A_i]}{\sum_{j=1}^n \Pr[A_j] \Pr[B A_j]}.$ |
| 26 | 67,108,864 | 101 | Normal (Gaussian) distribution: | Inclusion-exclusion: |
| 27 | 134,217,728 | 103 | $p(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2}, \quad E[X] = \mu.$ | $\Pr\left[\bigvee_{i=1}^n X_i\right] = \sum_{i=1}^n \Pr[X_i] +$ |
| 28 | 268,435,456 | 107 | The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is | $\sum_{k=2}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k} \Pr\left[\bigwedge_{j=1}^k X_{i_j}\right].$ |
| 29 | 536,870,912 | 109 | $nH_n.$ | Moment inequalities: |
| 30 | 1,073,741,824 | 113 | | $\Pr[X \geq \lambda E[X]] \leq \frac{1}{\lambda},$ |
| 31 | 2,147,483,648 | 127 | | $\Pr[X - E[X] \geq \lambda \cdot \sigma] \leq \frac{1}{\lambda^2}.$ |
| 32 | 4,294,967,296 | 131 | | Geometric distribution: |
| Pascal's Triangle | | | | $\Pr[X = k] = pq^{k-1}, \quad q = 1 - p,$ |
| 1 1 1 1 2 1 1 3 3 1 1 4 6 4 1 1 5 10 10 5 1 1 6 15 20 15 6 1 1 7 21 35 35 21 7 1 1 8 28 56 70 56 28 8 1 1 9 36 84 126 126 84 36 9 1 1 10 45 120 210 252 210 120 45 10 1 | | | | $E[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$ |

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Trigonometry



Pythagorean theorem:
 $C^2 = A^2 + B^2.$

Definitions:

$$\begin{aligned} \sin a &= A/C, & \cos a &= B/C, \\ \csc a &= C/A, & \sec a &= C/B, \\ \tan a &= \frac{\sin a}{\cos a} = \frac{A}{B}, & \cot a &= \frac{\cos a}{\sin a} = \frac{B}{A}. \end{aligned}$$

Area, radius of inscribed circle:

$$\frac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:

$$\sin x = \frac{1}{\csc x}, \quad \cos x = \frac{1}{\sec x},$$

$$\tan x = \frac{1}{\cot x}, \quad \sin^2 x + \cos^2 x = 1,$$

$$1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x,$$

$$\sin x = \cos\left(\frac{\pi}{2} - x\right), \quad \sin x = \sin(\pi - x),$$

$$\cos x = -\cos(\pi - x), \quad \tan x = \cot\left(\frac{\pi}{2} - x\right),$$

$$\cot x = -\cot(\pi - x), \quad \csc x = \cot\frac{x}{2} - \cot x,$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$

$$\sin 2x = 2 \sin x \cos x, \quad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$$

$$\cos 2x = \cos^2 x - \sin^2 x, \quad \cos 2x = 2 \cos^2 x - 1,$$

$$\cos 2x = 1 - 2 \sin^2 x, \quad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \quad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$$

$$\sin(x + y) \sin(x - y) = \sin^2 x - \sin^2 y,$$

$$\cos(x + y) \cos(x - y) = \cos^2 x - \sin^2 y.$$

Euler's equation:

$$e^{ix} = \cos x + i \sin x, \quad e^{i\pi} = -1.$$

Matrices

Multiplication:

$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^n a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.

$$\det A \cdot B = \det A \cdot \det B,$$

$$\det A = \sum_{\pi} \prod_{i=1}^n \text{sign}(\pi) a_{i,\pi(i)}.$$

2 × 2 and 3 × 3 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - fha - ibd.$$

Permanents:

$$\text{perm } A = \sum_{\pi} \prod_{i=1}^n a_{i,\pi(i)}.$$

Hyperbolic Functions

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \text{csch } x = \frac{1}{\sinh x},$$

$$\text{sech } x = \frac{1}{\cosh x}, \quad \text{coth } x = \frac{1}{\tanh x}.$$

Identities:

$$\cosh^2 x - \sinh^2 x = 1, \quad \tanh^2 x + \text{sech}^2 x = 1,$$

$$\text{coth}^2 x - \text{csch}^2 x = 1, \quad \sinh(-x) = -\sinh x,$$

$$\cosh(-x) = \cosh x, \quad \tanh(-x) = -\tanh x,$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y,$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y,$$

$$\sinh 2x = 2 \sinh x \cosh x,$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$

$$\cosh x + \sinh x = e^x, \quad \cosh x - \sinh x = e^{-x},$$

$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$

$$2 \sinh^2 \frac{x}{2} = \cosh x - 1, \quad 2 \cosh^2 \frac{x}{2} = \cosh x + 1.$$

| θ | $\sin \theta$ | $\cos \theta$ | $\tan \theta$ |
|----------|---------------|---------------|---------------|
|----------|---------------|---------------|---------------|

| | | | |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
|---|---|---|---|

| | | | |
|-----------------|---------------|----------------------|----------------------|
| $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ |
|-----------------|---------------|----------------------|----------------------|

| | | | |
|-----------------|----------------------|----------------------|---|
| $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 |
|-----------------|----------------------|----------------------|---|

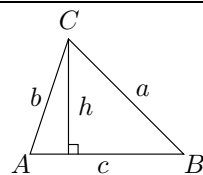
| | | | |
|-----------------|----------------------|---------------|------------|
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
|-----------------|----------------------|---------------|------------|

| | | | |
|-----------------|---|---|----------|
| $\frac{\pi}{2}$ | 1 | 0 | ∞ |
|-----------------|---|---|----------|

... in mathematics you don't understand things, you just get used to them.

– J. von Neumann

More Trig.



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Area:

$$\begin{aligned} A &= \frac{1}{2}hc, \\ &= \frac{1}{2}ab \sin C, \\ &= \frac{c^2 \sin A \sin B}{2 \sin C}. \end{aligned}$$

Heron's formula:

$$\begin{aligned} A &= \sqrt{s \cdot s_a \cdot s_b \cdot s_c}, \\ s &= \frac{1}{2}(a + b + c), \end{aligned}$$

$$s_a = s - a,$$

$$s_b = s - b,$$

$$s_c = s - c.$$

More identities:

$$\sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$

$$\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$

$$\tan \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$

$$= \frac{1 - \cos x}{\sin x},$$

$$= \frac{\sin x}{1 + \cos x},$$

$$\cot \frac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$

$$= \frac{1 + \cos x}{\sin x},$$

$$= \frac{\sin x}{1 - \cos x},$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$

$$\tan x = -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$

$$= -i \frac{e^{2ix} - 1}{e^{2ix} + 1},$$

$$\sin x = \frac{\sinh ix}{i},$$

$$\cos x = \cosh ix,$$

$$\tan x = \frac{\tanh ix}{i}.$$

Theoretical Computer Science Cheat Sheet

Number Theory

The Chinese remainder theorem: There exists a number C such that:

$$C \equiv r_1 \pmod{m_1}$$

\vdots
 \vdots
 \vdots

$$C \equiv r_n \pmod{m_n}$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x . If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^n p_i^{e_i-1} (p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \pmod{b}.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \pmod{p}.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^n \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: n is a prime iff

$$(n - 1)! \equiv -1 \pmod{n}.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of } \\ & r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n}$$

$$+ O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3}$$

$$+ O\left(\frac{n}{(\ln n)^4}\right).$$

Graph Theory

Definitions:

Loop An edge connecting a vertex to itself.

Directed Each edge has a direction.

Simple Graph with no loops or multi-edges.

Walk A sequence $v_0 e_1 v_1 \dots e_\ell v_\ell$.

Trail A walk with distinct edges.

Path A trail with distinct vertices.

Connected A graph where there exists a path between any two vertices.

Component A maximal connected subgraph.

Tree A connected acyclic graph.

Free tree A tree with no root.

DAG Directed acyclic graph.

Eulerian Graph with a trail visiting each edge exactly once.

Hamiltonian Graph with a cycle visiting each vertex exactly once.

Cut A set of edges whose removal increases the number of components.

Cut-set A minimal cut.

Cut edge A size 1 cut.

k-Connected A graph connected with the removal of any $k - 1$ vertices.

k-Tough $\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq |S|$.

k-Regular A graph where all vertices have degree k .

k-Factor A k -regular spanning subgraph.

Matching A set of edges, no two of which are adjacent.

Clique A set of vertices, all of which are adjacent.

Ind. set A set of vertices, none of which are adjacent.

Vertex cover A set of vertices which cover all edges.

Planar graph A graph which can be embedded in the plane.

Plane graph An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5 .

Notation:

$E(G)$ Edge set

$V(G)$ Vertex set

$c(G)$ Number of components

$G[S]$ Induced subgraph

$\deg(v)$ Degree of v

$\Delta(G)$ Maximum degree

$\delta(G)$ Minimum degree

$\chi(G)$ Chromatic number

$\chi_E(G)$ Edge chromatic number

G^c Complement graph

K_n Complete graph

K_{n_1, n_2} Complete bipartite graph

$r(k, \ell)$ Ramsey number

Geometry

Projective coordinates: triples (x, y, z) , not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian Projective

$$(x, y) \quad (x, y, 1)$$

$$y = mx + b \quad (m, -1, b)$$

$$x = c \quad (1, 0, -c)$$

Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

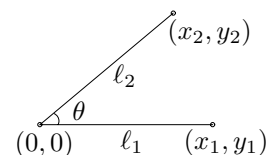
$$[|x_1 - x_0|^p + |y_1 - y_0|^p]^{1/p},$$

$$\lim_{p \rightarrow \infty} [|x_1 - x_0|^p + |y_1 - y_0|^p]^{1/p}.$$

Area of triangle (x_0, y_0) , (x_1, y_1) and (x_2, y_2) :

$$\frac{1}{2} \text{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos \theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{l_1 l_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1) :

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \quad V = \frac{4}{3} \pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.

– Issac Newton

Theoretical Computer Science Cheat Sheet

π

Wallis' identity:

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:

$$\frac{\pi}{4} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \cdots}}}}$$

Gregory's series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Newton's series:

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots \right)$$

Euler's series:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x . We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D , divide N by D , obtaining

$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$

where the degree of N' is less than that of D . Second, factor $D(x)$. Use the following rules: For a non-repeated factor:

$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$

where

$$A = \left[\frac{N(x)}{D(x)} \right]_{x=a}.$$

For a repeated factor:

$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$

where

$$A_k = \frac{1}{k!} \left[\frac{d^k}{dx^k} \left(\frac{N(x)}{D(x)} \right) \right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Calculus

Derivatives:

$$1. \frac{d(cu)}{dx} = c \frac{du}{dx}, \quad 2. \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}, \quad 3. \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx},$$

$$4. \frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx}, \quad 5. \frac{d(u/v)}{dx} = \frac{v \left(\frac{du}{dx} \right) - u \left(\frac{dv}{dx} \right)}{v^2}, \quad 6. \frac{d(e^{cu})}{dx} = ce^{cu} \frac{du}{dx},$$

$$7. \frac{d(c^u)}{dx} = (\ln c) c^u \frac{du}{dx}, \quad 8. \frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx},$$

$$9. \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}, \quad 10. \frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx},$$

$$11. \frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}, \quad 12. \frac{d(\cot u)}{dx} = \csc^2 u \frac{du}{dx},$$

$$13. \frac{d(\sec u)}{dx} = \tan u \sec u \frac{du}{dx}, \quad 14. \frac{d(\csc u)}{dx} = -\cot u \csc u \frac{du}{dx},$$

$$15. \frac{d(\arcsin u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad 16. \frac{d(\arccos u)}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx},$$

$$17. \frac{d(\arctan u)}{dx} = \frac{1}{1+u^2} \frac{du}{dx}, \quad 18. \frac{d(\operatorname{arccot} u)}{dx} = \frac{-1}{1+u^2} \frac{du}{dx},$$

$$19. \frac{d(\operatorname{arcsec} u)}{dx} = \frac{1}{u\sqrt{1-u^2}} \frac{du}{dx}, \quad 20. \frac{d(\operatorname{arccsc} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx},$$

$$21. \frac{d(\sinh u)}{dx} = \cosh u \frac{du}{dx}, \quad 22. \frac{d(\cosh u)}{dx} = \sinh u \frac{du}{dx},$$

$$23. \frac{d(\tanh u)}{dx} = \operatorname{sech}^2 u \frac{du}{dx}, \quad 24. \frac{d(\operatorname{coth} u)}{dx} = -\operatorname{csch}^2 u \frac{du}{dx},$$

$$25. \frac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \tanh u \frac{du}{dx}, \quad 26. \frac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \operatorname{coth} u \frac{du}{dx},$$

$$27. \frac{d(\operatorname{arcsinh} u)}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}, \quad 28. \frac{d(\operatorname{arccosh} u)}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx},$$

$$29. \frac{d(\operatorname{arctanh} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx}, \quad 30. \frac{d(\operatorname{arcoth} u)}{dx} = \frac{1}{u^2-1} \frac{du}{dx},$$

$$31. \frac{d(\operatorname{arcsech} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx}, \quad 32. \frac{d(\operatorname{arccsch} u)}{dx} = \frac{-1}{|u|\sqrt{1+u^2}} \frac{du}{dx}.$$

Integrals:

$$1. \int cu \, dx = c \int u \, dx, \quad 2. \int (u+v) \, dx = \int u \, dx + \int v \, dx,$$

$$3. \int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1, \quad 4. \int \frac{1}{x} \, dx = \ln x, \quad 5. \int e^x \, dx = e^x,$$

$$6. \int \frac{dx}{1+x^2} = \arctan x, \quad 7. \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx,$$

$$8. \int \sin x \, dx = -\cos x, \quad 9. \int \cos x \, dx = \sin x,$$

$$10. \int \tan x \, dx = -\ln |\cos x|, \quad 11. \int \cot x \, dx = \ln |\cos x|,$$

$$12. \int \sec x \, dx = \ln |\sec x + \tan x|, \quad 13. \int \csc x \, dx = \ln |\csc x + \cot x|,$$

$$14. \int \arcsin \frac{x}{a} \, dx = \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0,$$

Theoretical Computer Science Cheat Sheet

Calculus Cont.

15. $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$
16. $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$
17. $\int \sin^2(ax) dx = \frac{1}{2a} (ax - \sin(ax) \cos(ax)),$
18. $\int \cos^2(ax) dx = \frac{1}{2a} (ax + \sin(ax) \cos(ax)),$
19. $\int \sec^2 x dx = \tan x,$
20. $\int \csc^2 x dx = -\cot x,$
21. $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx,$
22. $\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx,$
23. $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx, \quad n \neq 1,$
24. $\int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx, \quad n \neq 1,$
25. $\int \sec^n x dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx, \quad n \neq 1,$
26. $\int \csc^n x dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx, \quad n \neq 1,$
27. $\int \sinh x dx = \cosh x,$
28. $\int \cosh x dx = \sinh x,$
29. $\int \tanh x dx = \ln |\cosh x|,$
30. $\int \coth x dx = \ln |\sinh x|,$
31. $\int \operatorname{sech} x dx = \arctan \sinh x,$
32. $\int \operatorname{csch} x dx = \ln \left| \tanh \frac{x}{2} \right|,$
33. $\int \sinh^2 x dx = \frac{1}{4} \sinh(2x) - \frac{1}{2} x,$
34. $\int \cosh^2 x dx = \frac{1}{4} \sinh(2x) + \frac{1}{2} x,$
35. $\int \operatorname{sech}^2 x dx = \tanh x,$
36. $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$
37. $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln |a^2 - x^2|,$
38. $\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \frac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \frac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$
39. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left(x + \sqrt{a^2 + x^2} \right), \quad a > 0,$
40. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$
41. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
42. $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8} (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
43. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$
44. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|,$
45. $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 - x^2}},$
46. $\int \sqrt{a^2 \pm x^2} dx = \frac{x}{2} \sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln \left| x + \sqrt{a^2 \pm x^2} \right|,$
47. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right|, \quad a > 0,$
48. $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln \left| \frac{x}{a+bx} \right|,$
49. $\int x \sqrt{a+bx} dx = \frac{2(3bx - 2a)(a+bx)^{3/2}}{15b^2},$
50. $\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{1}{x\sqrt{a+bx}} dx,$
51. $\int \frac{x}{\sqrt{a+bx}} dx = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right|, \quad a > 0,$
52. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
53. $\int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} (a^2 - x^2)^{3/2},$
54. $\int x^2 \sqrt{a^2 - x^2} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
55. $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
56. $\int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$
57. $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
58. $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right|,$
59. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$
60. $\int x \sqrt{x^2 \pm a^2} dx = \frac{1}{3} (x^2 \pm a^2)^{3/2},$
61. $\int \frac{dx}{x \sqrt{x^2 + a^2}} = \frac{1}{a} \ln \left| \frac{x}{a + \sqrt{a^2 + x^2}} \right|,$

Theoretical Computer Science Cheat Sheet

Calculus Cont.

62. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \arccos \frac{a}{|x|}, \quad a > 0,$ **63.** $\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x},$
64. $\int \frac{x dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2},$ **65.** $\int \frac{\sqrt{x^2 \pm a^2}}{x^4} dx = \mp \frac{(x^2 + a^2)^{3/2}}{3a^2 x^3},$
66. $\int \frac{dx}{ax^2 + bx + c} = \begin{cases} \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right|, & \text{if } b^2 > 4ac, \\ \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}}, & \text{if } b^2 < 4ac, \end{cases}$
67. $\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right|, & \text{if } a > 0, \\ \frac{1}{\sqrt{-a}} \arcsin \frac{-2ax - b}{\sqrt{b^2 - 4ac}}, & \text{if } a < 0, \end{cases}$
68. $\int \sqrt{ax^2 + bx + c} dx = \frac{2ax + b}{4a} \sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a} \int \frac{dx}{\sqrt{ax^2 + bx + c}},$
69. $\int \frac{x dx}{\sqrt{ax^2 + bx + c}} = \frac{\sqrt{ax^2 + bx + c}}{a} - \frac{b}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}},$
70. $\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{-1}{\sqrt{c}} \ln \left| \frac{2\sqrt{c}\sqrt{ax^2 + bx + c} + bx + 2c}{x} \right|, & \text{if } c > 0, \\ \frac{1}{\sqrt{-c}} \arcsin \frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}, & \text{if } c < 0, \end{cases}$
71. $\int x^3 \sqrt{x^2 + a^2} dx = \left(\frac{1}{3}x^2 - \frac{2}{15}a^2\right)(x^2 + a^2)^{3/2},$
72. $\int x^n \sin(ax) dx = -\frac{1}{a}x^n \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax) dx,$
73. $\int x^n \cos(ax) dx = \frac{1}{a}x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) dx,$
74. $\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx,$
75. $\int x^n \ln(ax) dx = x^{n+1} \left(\frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2} \right),$
76. $\int x^n (\ln ax)^m dx = \frac{x^{n+1}}{n+1} (\ln ax)^m - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} dx.$

Finite Calculus

Difference, shift operators:
 $\Delta f(x) = f(x+1) - f(x),$
 $E f(x) = f(x+1).$
 Fundamental Theorem:
 $f(x) = \Delta F(x) \Leftrightarrow \sum f(x) \delta x = F(x) + C.$

$$\sum_a^b f(x) \delta x = \sum_{i=a}^{b-1} f(i).$$

 Differences:
 $\Delta(cu) = c\Delta u, \quad \Delta(u+v) = \Delta u + \Delta v,$
 $\Delta(uv) = u\Delta v + E v \Delta u,$
 $\Delta(x^n) = nx^{n-1},$
 $\Delta(H_x) = x^{-1}, \quad \Delta(2^x) = 2^x,$
 $\Delta(c^x) = (c-1)c^x, \quad \Delta\binom{x}{m} = \binom{x}{m-1}.$
 Sums:
 $\sum cu \delta x = c \sum u \delta x,$
 $\sum (u+v) \delta x = \sum u \delta x + \sum v \delta x,$
 $\sum u \Delta v \delta x = uv - \sum E v \Delta u \delta x,$
 $\sum x^n \delta x = \frac{x^{n+1}}{n+1}, \quad \sum x^{-1} \delta x = H_x,$
 $\sum c^x \delta x = \frac{c^x}{c-1}, \quad \sum \binom{x}{m} \delta x = \binom{x}{m+1}.$
 Falling Factorial Powers:
 $x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n > 0,$
 $x^{\underline{0}} = 1,$
 $x^{\underline{n}} = \frac{1}{(x+1)\cdots(x+|n|)}, \quad n < 0,$
 $x^{\underline{n+m}} = x^{\underline{n}}(x-m)^{\underline{n}}.$
 Rising Factorial Powers:
 $x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n > 0,$
 $x^{\overline{0}} = 1,$
 $x^{\overline{n}} = \frac{1}{(x-1)\cdots(x-|n|)}, \quad n < 0,$
 $x^{\overline{n+m}} = x^{\overline{n}}(x+m)^{\overline{n}}.$

Conversion:

$x^{\underline{n}} = (-1)^n (-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$
 $= 1/(x+1)^{\overline{-n}},$
 $x^{\overline{n}} = (-1)^n (-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$
 $= 1/(x-1)^{\underline{-n}},$
 $x^n = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\underline{k}} = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (-1)^{n-k} x^{\overline{k}},$
 $x^{\underline{n}} = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] (-1)^{n-k} x^k,$
 $x^{\overline{n}} = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] x^k.$

| | | | |
|----------------------|---|-----------------------|--|
| $x^1 =$ | $x^{\underline{1}}$ | $=$ | $x^{\overline{1}}$ |
| $x^2 =$ | $x^{\underline{2}} + x^{\underline{1}}$ | $=$ | $x^{\overline{2}} - x^{\overline{1}}$ |
| $x^3 =$ | $x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}}$ | $=$ | $x^{\overline{3}} - 3x^{\overline{2}} + x^{\overline{1}}$ |
| $x^4 =$ | $x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}}$ | $=$ | $x^{\overline{4}} - 6x^{\overline{3}} + 7x^{\overline{2}} - x^{\overline{1}}$ |
| $x^5 =$ | $x^{\underline{5}} + 15x^{\underline{4}} + 25x^{\underline{3}} + 10x^{\underline{2}} + x^{\underline{1}}$ | $=$ | $x^{\overline{5}} - 15x^{\overline{4}} + 25x^{\overline{3}} - 10x^{\overline{2}} + x^{\overline{1}}$ |
| $x^{\overline{1}} =$ | x^1 | $x^{\underline{1}} =$ | x^1 |
| $x^{\overline{2}} =$ | $x^2 + x^1$ | $x^{\underline{2}} =$ | $x^2 - x^1$ |
| $x^{\overline{3}} =$ | $x^3 + 3x^2 + 2x^1$ | $x^{\underline{3}} =$ | $x^3 - 3x^2 + 2x^1$ |
| $x^{\overline{4}} =$ | $x^4 + 6x^3 + 11x^2 + 6x^1$ | $x^{\underline{4}} =$ | $x^4 - 6x^3 + 11x^2 - 6x^1$ |
| $x^{\overline{5}} =$ | $x^5 + 10x^4 + 35x^3 + 50x^2 + 24x^1$ | $x^{\underline{5}} =$ | $x^5 - 10x^4 + 35x^3 - 50x^2 + 24x^1$ |

Theoretical Computer Science Cheat Sheet

Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \dots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

| | | |
|---|---|--|
| $\frac{1}{1-x}$ | $= 1 + x + x^2 + x^3 + x^4 + \dots$ | $= \sum_{i=0}^{\infty} x^i,$ |
| $\frac{1}{1-cx}$ | $= 1 + cx + c^2x^2 + c^3x^3 + \dots$ | $= \sum_{i=0}^{\infty} c^i x^i,$ |
| $\frac{1}{1-x^n}$ | $= 1 + x^n + x^{2n} + x^{3n} + \dots$ | $= \sum_{i=0}^{\infty} x^{ni},$ |
| $\frac{x}{(1-x)^2}$ | $= x + 2x^2 + 3x^3 + 4x^4 + \dots$ | $= \sum_{i=0}^{\infty} ix^i,$ |
| $x^k \frac{d^n}{dx^n} \left(\frac{1}{1-x} \right)$ | $= x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \dots$ | $= \sum_{i=0}^{\infty} i^n x^i,$ |
| e^x | $= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$ | $= \sum_{i=0}^{\infty} \frac{x^i}{i!},$ |
| $\ln(1+x)$ | $= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$ | $= \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i},$ |
| $\ln \frac{1}{1-x}$ | $= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots$ | $= \sum_{i=1}^{\infty} \frac{x^i}{i},$ |
| $\sin x$ | $= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$ | $= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$ |
| $\cos x$ | $= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ | $= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$ |
| $\tan^{-1} x$ | $= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$ | $= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$ |
| $(1+x)^n$ | $= 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$ | $= \sum_{i=0}^{\infty} \binom{n}{i} x^i,$ |
| $\frac{1}{(1-x)^{n+1}}$ | $= 1 + (n+1)x + \binom{n+2}{2}x^2 + \dots$ | $= \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$ |
| $\frac{x}{e^x - 1}$ | $= 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \dots$ | $= \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$ |
| $\frac{1}{2x}(1 - \sqrt{1-4x})$ | $= 1 + x + 2x^2 + 5x^3 + \dots$ | $= \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{2i}{i} x^i,$ |
| $\frac{1}{\sqrt{1-4x}}$ | $= 1 + x + 2x^2 + 6x^3 + \dots$ | $= \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$ |
| $\frac{1}{\sqrt{1-4x}} \left(\frac{1 - \sqrt{1-4x}}{2x} \right)^n$ | $= 1 + (2+n)x + \binom{4+n}{2}x^2 + \dots$ | $= \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$ |
| $\frac{1}{1-x} \ln \frac{1}{1-x}$ | $= x + \frac{3}{2}x^2 + \frac{11}{6}x^3 + \frac{25}{12}x^4 + \dots$ | $= \sum_{i=1}^{\infty} H_i x^i,$ |
| $\frac{1}{2} \left(\ln \frac{1}{1-x} \right)^2$ | $= \frac{1}{2}x^2 + \frac{3}{4}x^3 + \frac{11}{24}x^4 + \dots$ | $= \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i},$ |
| $\frac{x}{1-x-x^2}$ | $= x + x^2 + 2x^3 + 3x^4 + \dots$ | $= \sum_{i=0}^{\infty} F_i x^i,$ |
| $\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2}$ | $= F_n x + F_{2n} x^2 + F_{3n} x^3 + \dots$ | $= \sum_{i=0}^{\infty} F_{ni} x^i.$ |

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{j^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$x A'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x) dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^i a_j$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^i a_j b_{i-j} \right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
- Leopold Kronecker

Theoretical Computer Science Cheat Sheet

Series

Expansions:

$$\frac{1}{(1-x)^{n+1}} \ln \frac{1}{1-x} = \sum_{i=0}^{\infty} (H_{n+i} - H_n) \binom{n+i}{i} x^i,$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

$$\left(\ln \frac{1}{1-x} \right)^n = \sum_{i=0}^{\infty} \binom{i}{n} \frac{n! x^i}{i!},$$

$$\tan x = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i} (2^{2i} - 1) B_{2i} x^{2i-1}}{(2i)!},$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty} \frac{\mu(i)}{i^x},$$

$$\zeta(x) = \prod_p \frac{1}{1-p^{-x}},$$

$$\zeta^2(x) = \sum_{i=1}^{\infty} \frac{d(i)}{x^i} \quad \text{where } d(n) = \sum_{d|n} 1,$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty} \frac{S(i)}{x^i} \quad \text{where } S(n) = \sum_{d|n} d,$$

$$\zeta(2n) = \frac{2^{2n-1} |B_{2n}| \pi^{2n}}{(2n)!}, \quad n \in \mathbb{N},$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty} (-1)^{i-1} \frac{(4^i - 2) B_{2i} x^{2i}}{(2i)!},$$

$$\left(\frac{1 - \sqrt{1-4x}}{2x} \right)^n = \sum_{i=0}^{\infty} \frac{n(2i+n-1)!}{i!(n+i)!} x^i,$$

$$e^x \sin x = \sum_{i=1}^{\infty} \frac{2^{i/2} \sin \frac{i\pi}{4}}{i!} x^i,$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty} \frac{(4i)!}{16^i \sqrt{2} (2i)!(2i+1)!} x^i,$$

$$\left(\frac{\arcsin x}{x} \right)^2 = \sum_{i=0}^{\infty} \frac{4^i i!^2}{(i+1)(2i+1)!} x^{2i}.$$

$$\left(\frac{1}{x} \right)^{\overline{-n}} = \sum_{i=0}^{\infty} \left\{ \begin{matrix} i \\ n \end{matrix} \right\} x^i,$$

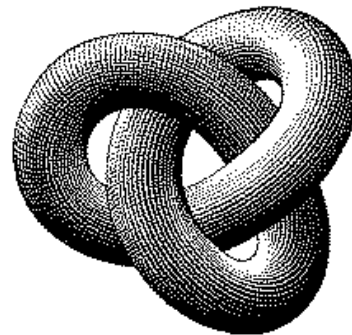
$$(e^x - 1)^n = \sum_{i=0}^{\infty} \left\{ \begin{matrix} i \\ n \end{matrix} \right\} \frac{n! x^i}{i!},$$

$$x \cot x = \sum_{i=0}^{\infty} \frac{(-4)^i B_{2i} x^{2i}}{(2i)!},$$

$$\zeta(x) = \sum_{i=1}^{\infty} \frac{1}{i^x},$$

$$\frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty} \frac{\phi(i)}{i^x},$$

Escher's Knot



Stieltjes Integration

If G is continuous in the interval $[a, b]$ and F is nondecreasing then

$$\int_a^b G(x) dF(x)$$

exists. If $a \leq b \leq c$ then

$$\int_a^c G(x) dF(x) = \int_a^b G(x) dF(x) + \int_b^c G(x) dF(x).$$

If the integrals involved exist

$$\int_a^b (G(x) + H(x)) dF(x) = \int_a^b G(x) dF(x) + \int_a^b H(x) dF(x),$$

$$\int_a^b G(x) d(F(x) + H(x)) = \int_a^b G(x) dF(x) + \int_a^b G(x) dH(x),$$

$$\int_a^b c \cdot G(x) dF(x) = \int_a^b G(x) d(c \cdot F(x)) = c \int_a^b G(x) dF(x),$$

$$\int_a^b G(x) dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x) dG(x).$$

If the integrals involved exist, and F possesses a derivative F' at every point in $[a, b]$ then

$$\int_a^b G(x) dF(x) = \int_a^b G(x) F'(x) dx.$$

Cramer's Rule

If we have equations:

$$a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n = b_2$$

$$\vdots \quad \quad \quad \vdots$$

$$a_{n,1}x_1 + a_{n,2}x_2 + \dots + a_{n,n}x_n = b_n$$

Let $A = (a_{i,j})$ and B be the column matrix (b_i) . Then there is a unique solution iff $\det A \neq 0$. Let A_i be A with column i replaced by B . Then

$$x_i = \frac{\det A_i}{\det A}.$$

| | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|
| 00 | 47 | 18 | 76 | 29 | 93 | 85 | 34 | 61 | 52 |
| 86 | 11 | 57 | 28 | 70 | 39 | 94 | 45 | 02 | 63 |
| 95 | 80 | 22 | 67 | 38 | 71 | 49 | 56 | 13 | 04 |
| 59 | 96 | 81 | 33 | 07 | 48 | 72 | 60 | 24 | 15 |
| 73 | 69 | 90 | 82 | 44 | 17 | 58 | 01 | 35 | 26 |
| 68 | 74 | 09 | 91 | 83 | 55 | 27 | 12 | 46 | 30 |
| 37 | 08 | 75 | 19 | 92 | 84 | 66 | 23 | 50 | 41 |
| 14 | 25 | 36 | 40 | 51 | 62 | 03 | 77 | 88 | 99 |
| 21 | 32 | 43 | 54 | 65 | 06 | 10 | 89 | 97 | 78 |
| 42 | 53 | 64 | 05 | 16 | 20 | 31 | 98 | 79 | 87 |

The Fibonacci number system:
Every integer n has a unique representation

$$n = F_{k_1} + F_{k_2} + \dots + F_{k_m},$$

where $k_i \geq k_{i+1} + 2$ for all i ,
 $1 \leq i < m$ and $k_m \geq 2$.

Fibonacci Numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Definitions:

$$F_i = F_{i-1} + F_{i-2}, \quad F_0 = F_1 = 1,$$

$$F_{-i} = (-1)^{i-1} F_i,$$

$$F_i = \frac{1}{\sqrt{5}} \left(\phi^i - \hat{\phi}^i \right),$$

Cassini's identity: for $i > 0$:

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n,$$

$$F_{2n} = F_n F_{n+1} + F_{n-1} F_n.$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$$

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
- William Blake (The Marriage of Heaven and Hell)